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Markov processes on the adeles

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1 Introduction

In this article we will construct Markov processes on the ring of adeles. We will give an investigation to the processes whose p-components are all semistable, and examine exiting times for them from some regions. It will be shown that the expectations concerning the exiting times represent Riemann's zeta function in the region of convergence of the Euler product.

For every prime integer p, we denote by $\mathbf{Q}_p$ the field of p-adic numbers and by $\mathbf{Z}_p$ its integer ring. The p-adic norm $|\ |_p$ is normalized so that $|p|_p = p^{-1}$.

2 Semistable processes on the p-adic fields

Since the p-adic field $\mathbf{Q}_p$ is totally disconnected, every stochastic process thereon is purely of jump type. Therefore an additive process $\{X_t\}_{t\geq 0}$ on $\mathbf{Q}_p$ is completely determined by its Lévy measure F. In [Y] it is shown that if the law of $\{X_t\}_{t\geq 0}$ is invariant by rotations around the origin, then the Lévy measure F, and hence the law of $\{X_t\}_{t\geq 0}$ corresponds in one-to-one way to a sequence

$\{a^{(p)}(m)\}_{m\in\mathbf{Z}}$ of positive numbers such that

$$a^{(p)}(m+1) \le a^{(p)}(m) \text{ for any } m \in \mathbf{Z}, \tag{1}$$

and

$$\lim_{m\to\infty} a^{(p)}(m) = 0. \tag{2}$$

3 Markov processes on the adeles

Let A be the ring of adeles on $\mathbf{Q}$, namely the ristricted product of $\mathbf{R} = \mathbf{Q}_\infty$ and $\mathbf{Q}_p$ for all prime integers p relative to their integer rings. We put S for the ristricted product of $\mathbf{Q}_p$ for all (finite) primes :

$$S := \left\{ (x_p)_p \in \prod_{p<\infty} \mathbf{Q}_p \right. \\ \left. : \ x_p \in \mathbf{Z}_p \text{ except for a finite number of } p \right\},$$

then we have $A = \mathbf{R} \times S$.

Let us fix an integer m. For each prime integer p, let $\{B_i^{(p,m)}\}_{i=0,1,2,\ldots}$ be the set of disjoint p-adic discs of radius p^m such that $\mathbf{Q}_p = \cup_{i=0}^{\infty} B_i^{(p,m)}$. If we put $\mathbf{B}_{\mathbf{i}}^{(m)} = \prod_p B_{i_p}^{(p,m)}$ for every sequence $\mathbf{i} = (i_p)_p$ indexed by prime integers p, then the direct product space $\prod_{p<\infty} \mathbf{Q}_p$ is the disjoint union of the sets $\mathbf{B}_{\mathbf{i}}^{(m)}$.

Take a sequence $\{a^{(p)}(m)\}_{m\in\mathbf{Z}}$ for each prime p satisfying (1), (2), as well as

$$0 < \sum_p a^{(p)}(m) < \infty, \tag{3}$$

and for non-negative integers i and j, define $a_{ij}^{(p)}(m)$ by

$$a_{ij}^{(p)}(m) := \begin{cases} -a^{(p)}(m), & \text{if } i = j, \\ (p-1)^{-1}p^{-k+1}\left(a^{(p)}(m+k-1) - a^{(p)}(m+k)\right), & \\ & \text{if } d_p\left(B_i^{(p,m)}, B_j^{(p,m)}\right) = p^{m+k}. \end{cases}$$

Here for $i \neq j$, $d_p\left(B_i^{(p,m)}, B_j^{(p,m)}\right)$ is the p-adic distance between the balls $B_i^{(p,m)}$ and $B_j^{(p,m)}$, which is well-defined by $|x_i - x_j|_p$ for arbitrarily chosen points $x_i \in B_i^{(p,m)}$ and $x_j \in B_j^{(p,m)}$. Then for sequences $\mathbf{i} = (i_p)_p$ and $\mathbf{j} = (j_p)_p$ we define $c_{\mathbf{ij}}(m)$ by

$$c_{\mathbf{ij}}(m) := \begin{cases} \sum_p a_{i_p i_p}^{(p)}(m), & \text{if } i_p = j_p \text{ for all } p, \\ a_{i_p j_p}^{(p)}(m), & \text{if } i_p \neq j_p, \text{ and if } i_{p'} = j_{p'} \text{ for all } p' \neq p, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the Kolmogorov's differential equations

$$\begin{cases} P_{\mathbf{ij}}^{(m)\prime}(t) = \sum_{\mathbf{k}} c_{\mathbf{kj}}(m) P_{\mathbf{ik}}^{(m)}(t), \\ P_{\mathbf{ij}}^{(m)\prime}(t) = \sum_{\mathbf{k}} c_{\mathbf{ik}}(m) P_{\mathbf{kj}}^{(m)}(t). \end{cases}$$

Then it can be seen that there exists a unique solution $\left\{P_{\mathbf{ij}}^{(m)}(t)\right\}_{t \geq 0}$ satisfying the followings :

$$\begin{aligned} & P_{\mathbf{ij}}^{(m)}(t) \geq 0, \\ & P_{\mathbf{ij}}^{(m)}(0) = \delta_{\mathbf{ij}}, \\ & \sum_{\mathbf{j}} P_{\mathbf{ij}}^{(m)}(t) = 1 \quad \text{for all } \mathbf{i}, \\ & P_{\mathbf{ij}}^{(m)}(t+s) = \sum_{\mathbf{k}} P_{\mathbf{ik}}^{(m)}(t) P_{\mathbf{kj}}^{(m)}(s). \end{aligned}$$

The solution $\left\{P_{\mathbf{ij}}^{(m)}(t)\right\}_{t\geq 0}$ gives a Markovian semigroup on the set of sequences $\mathbf{i} = (i_p)_p$, and by identifying the sequence $\mathbf{i}$ with the subset $\mathbf{B}_{\mathbf{i}}^{(m)}$ in the direct product space $\Pi_{p<\infty} \mathbf{Q}_p$, we obtain a Markov chain on $\left\{\mathbf{B}_{\mathbf{i}}^{(m)}\right\}_{\mathbf{i}}$.

It is not hard to see that the semigroup $\left\{P_{\mathbf{ij}}^{(m)}(t)\right\}_{t\geq 0}$ is consistent with respect to the index m in the following sense. Let $m > m'$, and for each sequence $\mathbf{i}$, let $\mathcal{I}(\mathbf{i})$ be the set of all sequences $\mathbf{i}'$ such that $\mathbf{B}_{\mathbf{i}'}^{(m')} \subset \mathbf{B}_{\mathbf{i}}^{(m)}$. Then $\mathbf{B}_{\mathbf{i}}^{(m)}$ is the disjoint union of $\mathbf{B}_{\mathbf{i}'}^{(m')}$ for $\mathbf{i}' \in \mathcal{I}(\mathbf{i})$, and it follows that

$$P_{\mathbf{ij}}^{(m)}(t) = \sum_{\mathbf{j}'\in\mathcal{I}(\mathbf{j})} P_{\mathbf{i}'\mathbf{j}'}^{(m')}(t),$$

for any $\mathbf{i}' \in \mathcal{I}(\mathbf{i})$.

By this consistency, we can define a Markovian semigroup $\{\tilde{P}_t(\mathbf{x}, \cdot)\}$ on the direct product space $\Pi_{p<\infty} \mathbf{Q}_p$ by

$$\tilde{P}_t\left(\mathbf{x}, \mathbf{B}_{\mathbf{j}}^{(m)}\right) = P_{\mathbf{ij}}^{(m)}(t),$$

where $\mathbf{i}$ is taken so that $\mathbf{x} \in \mathbf{B}_{\mathbf{i}}^{(m)}$.

We have constructed a semigroup $\{\tilde{P}_t(\mathbf{x}, \cdot)\}$ on the direct product space $\Pi_{p<\infty} \mathbf{Q}_p$, whereas we have the following.

Proposition 3.1. *For any* $\mathbf{x}_0 \in S$ *and* $t \geq 0$, *we have* $\tilde{P}_t(\mathbf{x}_0, S) = 1$.

If we take any Markovian semigroup $\{\mu_t\}_{t\geq 0}$ on $\mathbf{R}$, then by this proposition $\{P_t := \mu_t * \tilde{P}_t\}_{t\geq 0}$ gives a Markovian semigroup on the adele ring A. Let

$$\mathbf{X}_t = (X_t^{(\infty)}, X_t^{(2)}, X_t^{(3)}, \ldots, X_t^{(p)}, \ldots)$$

be the corresponding Markov process on A.

4 Semistable processes on the adeles and representation of Riemann's zeta function

We examine the particular case where for every prime p the sequence $\{a^{(p)}(m)\}_{m\in\mathbf{Z}}$ is a geometric sequence :

$$a^{(p)}(m) = c_p p^{-\alpha_p m}$$

with some positive numbers c_p and α_p. Then as is proved in [Y], the p-th component $X_t^{(p)}$ is a semistable process on $\mathbf{Q}_p$ satisfying $X_{p^{-\alpha_p}t}^{(p)} = pX_t^{(p)}$ in law. Let τ_p be the first exiting time of $X_t^{(p)}$ from $p\mathbf{Z}_p$:

$$\tau_p := \inf\left\{t > 0 \ : \ X^{(p)}(t) \notin p\mathbf{Z}_p\right\},$$

and for a complex number s, define a random variable Y_s by

$$Y_s := \prod_{p<\infty} \left(1 - p^{-\alpha_p}\right)^{-1} \left|X_{\tau_p}^{(p)}\right|_p^{\alpha_p - s}.$$

Then we have the following.

Proposition 4.1. *If* $\mathrm{Re}(s) > 1$ *then* Y_s *has a finite expectation, and* $E[Y_s] = \zeta(s)$*, where* ζ *is Riemann's zeta function.*

We can apply this proposition with $\alpha_p = \mathrm{Re}(s)$ to obtain the following corollary.

Corollary 4.2. *For* $\mathrm{Re}(s) > 1$*, we have*

$$\frac{\zeta(s)}{\zeta(\mathrm{Re}(s))} = E\left[\prod_{p<\infty} \left|X_{\tau_p}^{(p)}\right|^{-\sqrt{-1}\mathrm{Im}(s)}\right],$$

$$\frac{\zeta(s)}{\zeta\left(\bar{s}\right)} = \frac{E\left[\prod_{p<\infty} \left|X_{\tau_p}^{(p)}\right|^{-\sqrt{-1}\mathrm{Im}(s)}\right]}{E\left[\prod_{p<\infty} \left|X_{\tau_p}^{(p)}\right|^{\sqrt{-1}\mathrm{Im}(s)}\right]},$$

where $\bar{s}$ denotes the conjugate complex number of s.

Let c be a constant with $0 < c < 1$, and take $\alpha_p = -\log_p c$ for all primes p. In this case the assumption (3) is equivalent to

$$0 < \sum_{p<\infty} c_p < \infty. \tag{4}$$

For a real number $s > 1$, take $c_p = -\log(1 - p^{-s})$, then they satisfy (4). Let us take any semistable process $X_t^{(\infty)}$ on $\mathbf{R}$ satisfying $X_{ct}^{(\infty)} = rX_t^{(\infty)}$ for some $r \in \mathbf{R}$, and let $\mathbf{X}_t = (X_t^{\infty}, X_t^{(2)}, X_t^{(3)}, \ldots, X_t^{(p)}, \ldots)$ be the Markov process on A as above. Then we can see that $\mathbf{X}_t$ becomes a semistable process on A as follows.

Proposition 4.3. *Let $\gamma = (r, 2, 3, \ldots, p, \ldots) \in A$, then it follows that $\mathbf{X}_{ct} = \gamma \mathbf{X}_t$.*

For this semistable process $\mathbf{X}_t$, let τ be the first exiting time from the set $\mathbf{R} \times \prod_{p<\infty} \mathbf{Z}_p$:

$$\tau := \inf\left\{t > 0 \ : \ \mathbf{X}_t \notin \mathbf{R} \times \prod_{p<\infty} \mathbf{Z}_p\right\}.$$

Then we have :

Proposition 4.4.

$$\zeta(s) = \exp E[\tau]^{-1}.$$

References

[Y] K. Yasuda : *Additive processes on local fields*, J. Math. Sci. Univ. Tokyo, **3**(1996), 629–654.